e-Pathshala पाठशाला

A Gateway to All Post Graduate Courses

MHRD Govt. of India

An MHRD Project under its National Mission on Education throught ICT (NME-ICT)

Subject: Physics

Production of Courseware

-Content for Post Graduate Courses

Paper No. : Classical Mechanics

Module **: Special Theory of Relativity**

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Writer	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Reviewer	

Description of Module	
Subject Name	Physics
Paper Name	Classical Mechanics
Module Name/Title	**Special Theory of Relativity**
Module Id	25

Special Theory of Relativity

Contents:

Learning Objectives :

❖ You will learn about the contravariant and covariant vectors and their transformation properties.

❖ You will learn to derive relativistic kinematic relations between energy, momentum and scattering angles for two particle scattering processes.

1. Introduction

In Newtonian world view space and time are independent absolute entities. The Newton's Laws of motion are invariant under Galilean transformations when we go from one inertial frame to another which is moving with a uniform velocity with respect to the one another. The Galilean transformations are given by $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t$; $t' = t$. $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the position vectors measured in the unprimed and primed initial frames which are moving with a relative velocity $\boldsymbol{v}$ with respect to each other. In Newtonian frame work time is independent of the reference frame. Maxwell's equations of electro–magnetism however, do not satisfy the principle of Galilean relativity. Maxwell's equations predicted that the speed of light in vacuum is a universal constant and is equal to C. If this is true in one coordinate system, it clearly will not be true in another coordinate system moving with a uniform velocity and defined by the Gallilean transformations. The pioneering attempts for the resolution of the problem of relativity in electrodynamics and mechanics were made by Lorentz and Poincare. The problem was finally resolved by Einstein in 1905 in the form of Postulates of Special Theory of Relativity namely,

(i) The Laws of physics are invariant in all inertial frames.

(ii) The velocity of light in free space is a universal constant independent of the frame of reference.

In Einstein's world view space and time are still absolute entities but they are not independent. They form a four–dimensional Minkowski space.

2. Lorentz Transformations

Lorentz transformations have the fundamental property of keeping the velocity of light constant in all inertial frames which means that

$$ds^2 = c^2dt^2 - d\boldsymbol{x}^2 \tag{25.1}$$

remains invariant under the transformations

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(t - \frac{\beta}{c}x\right) \end{aligned} \tag{25.2}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}; \qquad \beta = \frac{v}{c}$$

The Lorentz transformation is thus a transformation from one system of space–time coordinate x^μ to another system x'^μ such that

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{25.3}$$

where a^μ and Λ^μ_ν are constant. We define the coordinates. a^μ in a four–dimensional Minkowski space as

$$x^\mu = (x_0, \boldsymbol{x}) \tag{25.4}$$

We also distinguish the 'upper indices' (contravariant) and the lower (covariant indices) as

$$x_\mu = (x_0 - \boldsymbol{x}) \tag{25.5}$$

Which can be obtained from the contravariant coordinates by transformation through the matrix $g_{\mu\nu}$ namely

$$x_\mu = g_{\mu\nu} x^\nu \tag{25.6}$$

The scalar product

$$x^\mu x_\mu = x_0^2 - \boldsymbol{x}^2 = g_{\mu\nu} x^\nu x^\gamma \tag{25.7}$$

In terms of the matrix tensor

$$dS^2 = g_{\mu\nu} dx^\mu x^\nu \tag{25.8}$$

The matrix tensor is defined as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{25.9}$$

i.e.

$$\begin{aligned} g_{\mu\nu} &= +1 \quad \text{for} \quad \mu = \nu = 0 \\ &= -1 \quad \text{for} \quad \mu = \nu = 1,2,3 \\ &= -1 \quad \text{for} \quad \mu \neq \nu \end{aligned}$$

Lorentz transformation takes dx^μ to dx'^μ given by

$$dx'^\mu = \Lambda^\mu_\nu \; dx^\nu$$

$$dS'^2 = g_{\mu\nu} dx'^\mu \; dx'^\nu = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta \; dx^\alpha \; dx^\beta \tag{25.10}$$

$$= dS^2 = g_{\alpha\beta} \; dx^\alpha \; dx^\beta$$

The Lorentz transformations thus have the fundamental property

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu} = g_{\alpha\beta} \tag{25.11}$$

The matrix tensor has the following property:

$$i)\quad \det g = -1$$
$$ii)\quad g^{-1} = g^T = g$$
$$iii)\quad g_{\alpha\beta} = g^{\alpha\beta}$$
$$iv)\quad g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu$$
$$(Kronecker\ delta\ \delta_\mu^\nu = 1\ \ for\ \mu = \nu, = 0\ \ for\ \ \mu \neq \nu$$

We now define a four vector V^μ any quantity that transforms like dx^μ i.e

$$V^\mu \to V'^\mu = \Lambda_\nu^\mu\, V^\nu \tag{25.12}$$

When the coordinate system is transformed by

$$x^\mu \to x'^\mu = \Lambda_\nu^\mu\, x^\nu \tag{25.13}$$

V^μ like x^μ is called a contravariant four-order to distinguish it from the covariant vector V_μ which transforms as

$$V_\mu \to V'_\mu = \Lambda_\mu^\nu\, V_\nu \tag{25.14}$$

$$\text{Where}\quad \Lambda_\mu^\nu = g_{\mu\alpha}\ \ g^{\nu\beta}\ \ \Lambda_\beta^\alpha \tag{25.15}$$

The scalar product of a contravariant vector with a covariant vector is invariant under Lorentz transformations :

$$\begin{aligned} U'_\mu V'^\mu &= g^{\mu\alpha} U'_\mu\, V'_\alpha \\ &= g^{\mu\alpha}\ \Lambda_\mu^\nu\ \ \Lambda_\alpha^\beta U_\nu V_\beta \\ &= g^{\nu\beta}\ \ U_\nu V_\beta \\ &= \ U_\nu\, V^{\nu'} \end{aligned} \tag{25.16}$$

where we have used the property (25.11)

we define a four gradient operator:

$$\begin{aligned} \frac{\partial}{\partial x_\mu} &\equiv \left(\frac{\partial}{\partial x_0}, -\nabla\right) \\ &= \left(\frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right) \end{aligned} \tag{25.17}$$

The partial derivative with respect to the transformed coordinates x'^μ is given by

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x_\nu} \tag{25.18}$$

and we see that differentiation w.r.t. contravariant component of the coordinate vector transforms as a component of a covariant vector operator.

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, -\nabla\right) \quad (25.19)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x_0}, \nabla\right)$$

and the divergence of a four vector V^μ is invariant

$$\partial^\mu V_\mu = \partial_\mu V^\mu = \frac{\partial V^0}{\partial x_0} + \nabla \cdot \nabla \quad (25.20)$$

The four – dimensional Laplace operator is written as $\Box$

$$\boxdot \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial x_0^2} - \nabla^2 \quad (25.21)$$

3. Particle Dynamics

Let us consider a particle moving in a force field. We define the relativistic force f^μ acting on a particle with coordinate $x^\mu(\tau)$ by

$$f^\mu = m\frac{d^2 x^\mu}{d\tau^2} \quad (25.22)$$

Where $d\tau$ is the proper time $d\tau = (dt^2 - d\boldsymbol{x}^2)^{1/2}$ and we have put $c = 1$ in the natural units. Clearly, knowing f^μ we can compute the motion of the particle. The relativistic form (25.22) of Newton's second law allows us to define an energy–momentum four–vector

$$p^\mu \equiv m\frac{dx^\mu}{d\tau} \quad (25.23)$$

And the second law can be written as

$$\frac{dp^\mu}{d\tau} = f^\mu \quad (25.24)$$

We have $d\tau = (dt^2 - d\boldsymbol{x}^2)^{1/2} = (1 - \boldsymbol{V}^2)^{1/2} dt$

Where

$$\boldsymbol{V} = \frac{d\boldsymbol{x}}{dt}$$

Space components of p^μ form the momentum vector

$$\boldsymbol{p} = m\frac{d\boldsymbol{x}}{d\tau} = m\frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} = m\gamma\boldsymbol{v} \quad (25.25)$$

and the time component is the energy

$$p^0 \equiv E = m\frac{dt}{d\tau} = m\gamma \quad (25.26)$$

p^{μ} transforms like a four vector because m and $d\tau$ are invariants and dx^{μ} is a four–vector.

$$p'^{\mu} = \Lambda^{\mu}_{\nu}\, p^{\nu} \qquad (25.27)$$

From (25.25) and (25.26), we can eliminate the velocity to get the relation

$$E^2 = (\boldsymbol{p}^2 + m^2)$$

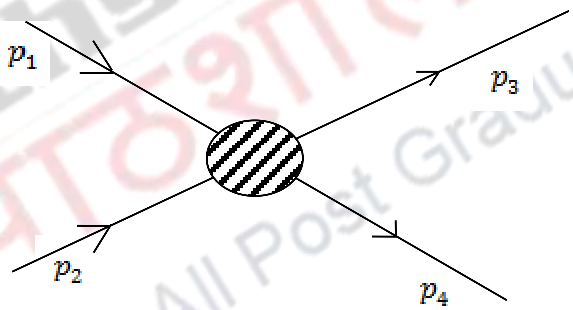
And the scalar product of $p^{\mu}p_{\mu}$ written as p^2 is given by

$$p^2 = p^{\mu}p_{\mu} = g_{\mu\nu}\, p^{\mu}\, p^{\nu} = E^2 - \boldsymbol{p}^2 = m^2 \qquad (25.28)$$

4. Relativistic Kinematics in Scattering

We consider two particle scattering

$$1 + 2 \rightarrow 3 + 4 \qquad (25.29)$$



The particles 1 – 4 has their four momenta $p_1^{\mu} - p_4^{\mu}$ and masses $m_1 - m_4$ respectively.

The law of conservation of energy and momentum gives:

$$p_1^{\mu} + p_2^{\mu} = p_3^{\mu} + p_4^{\mu} \qquad (25.30)$$

In terms of energy and momenta, we have

$$E_1 + E_2 = E_3 + E_4$$
$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4 \qquad (25.31)$$

The scalar products $p_i^{\mu}p_{i\mu} = p_i^2$ are

$$p_1^2 = m_1^2 \;\; ; \;\; p_2^2 = m_2^2 \;\; ; \;\; p_3^2 = m_3^2 \;\; ; p_4^2 = m_4^2 \qquad (25.32)$$

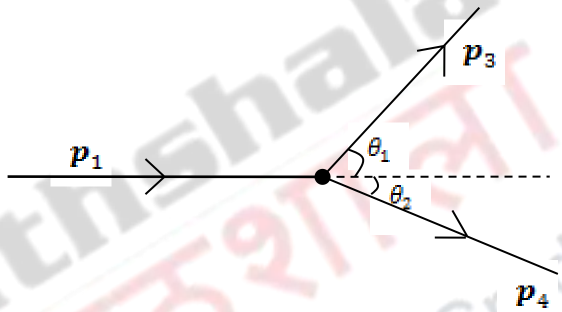
In order to express scalar products like

$p_i^{\mu}p_{j\mu} = p_i p_j (i \neq j)$ we use invariant Mendelstam variables s, t and u defined as

$$S=(p_1+p_2)^2=(p_3+p_4)^2$$
$$t=(p_1-p_3)^2=(p_2-p_4)^2 \qquad (25.33)$$
$$u=(p_1-p_4)^2=(p_2-p_3)^2$$

The three variables S, t and u are not independent, in fact

$$S+t+u=(p_1+p_2)^2+(p_1-p_3)^2+(p_1-p_4)^2$$
$$=3m_1^2+m_2^2+m_3^2+m_4^2+p_1\cdot(p_2-p_3-p_4)$$
$$=3m_1^2+m_2^2+m_3^2+m_4^2+2p_1\cdot(-p_1)$$
$$S+t+u=m_1^2+m_2^2+m_3^2+m_4^2 \qquad (25.34)$$



Laboratory Frame

In the Lab frame we have a particle 1 scattering on a target particle 2 which is at rest.

In the Lab $\qquad p_1^\mu=(E_1,\boldsymbol{p}_1)$ and $p_2^\mu=(m_2,\boldsymbol{o})$

$$p_3^\mu=(E_3,\boldsymbol{p}_3) \text{ and } p_4^\mu=(E_4,\boldsymbol{p}_4)$$

$$S=(p_1+p_2)^2=(E_1+m_2)^2-\boldsymbol{p}_1^2=m_1^2+m_2^2+2E_1m_2$$

$$\therefore \qquad E_1=\frac{S-m_1^2-m_2^2}{2m_2} \qquad (25.35)$$

$$t=(p_1-p_3)^2=m_1^2+m_3^2+2p_1p_3$$
$$=m_1^2+m_3^2-2E_1E_3+2|\boldsymbol{p}_1||\boldsymbol{p}_3|\cos\theta_1 \qquad (25.36)$$

$$|\boldsymbol{p}_1|^2=E_1^2-m_1^2=\left(\frac{S-m_1^2-m_2^2}{2m_2}\right)^2-m_1^2$$
$$=\frac{\lambda(S,m_1^2,m_2^2)}{4m_2^2}$$

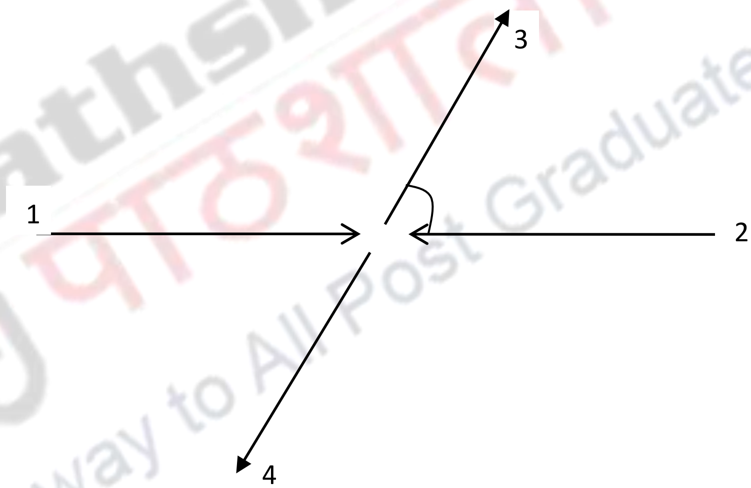
$$|\boldsymbol{p}_1|^2 = \frac{\lambda^{1/2}(S, m_1^2, m_2^2)}{2m_2} \tag{25.37}$$

Where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ (25.38)

$$u = (p_1 - p_4)^2 = m_1^2 + m_4^2 - 2E_1E_4 + 2|\boldsymbol{p}_1||\boldsymbol{p}_4|\cos\theta_2 \tag{25.39}$$

Centre of Mass (CM) Frame

It is more convenient to consider a scattering process in the CM frame, in which the CM is at rest and therefore the particles 1 and 2 approach each other with equal and opposite momenta and after scattering travel in opposite directions with equal momenta.



$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$$

$$p_1^\mu = (E_1, \boldsymbol{p}) \;, \qquad p_2^\mu = (E_1, -\boldsymbol{p})$$

$$p_3^\mu = (E_1, \boldsymbol{p}') \;, \qquad p_4^\mu = (E_1, -\boldsymbol{p}')$$

$$S = (p_1 + p_2)^2 = (E_1 + E_2)^2 = (E_1 + E_2)^2 \equiv E_{CM}^2 \tag{25.40}$$

$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2|\boldsymbol{p}_2||\boldsymbol{p}'|\cos\theta \tag{25.41}$$

$$= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2E_2E_4 + 2|\boldsymbol{p}||\boldsymbol{p}'|\cos\theta$$

Now

$$S = E_1^2 + E_2^2 + 2E_1 m_2$$

$$= (|\boldsymbol{p}|^2 + m_1^2) + (|\boldsymbol{p}|^2 + m_2^2) + 2\sqrt{|\boldsymbol{p}|^2 + m_1^2}\sqrt{|\boldsymbol{p}|^2 + m_2^2}$$

We get

$$|\boldsymbol{p}| = \frac{\lambda^{1/2}(S, m_1^2, m_2^2)}{2\sqrt{S}} \tag{25.42}$$

likewise

$$|\boldsymbol{p}'| = \frac{\lambda^{1/2}(S, m_3^2, m_4^2)}{2\sqrt{S}} \tag{25.43}$$

We also get

$$\begin{aligned} E_1 &= \frac{S + m_1^2 - m_2^2}{2\sqrt{S}} \\ E_2 &= \frac{S + m_2^2 - m_1^2}{2\sqrt{S}} \\ E_3 &= \frac{S + m_3^2 - m_4^2}{2\sqrt{S}} \\ E_4 &= \frac{S + m_4^2 - m_3^2}{2\sqrt{S}} \end{aligned} \tag{25.44}$$

For the case of elastic scattering $m_1 = m_3 \,;\, m_2 = m_4$

$$|\boldsymbol{p}| = |\boldsymbol{p}'| \;, \quad E_1 = E_3 \;, \quad E_2 = E_4$$

$$t = -2|\boldsymbol{p}|^2(1 - \cos\theta) = -4p^3 \sin^2\frac{\theta}{2}$$

t is also the –ve of the square of the momentum transfer since

$$t = (p_2 - p_4)^2 = -(\boldsymbol{p}_2 - \boldsymbol{p}_4)^2 = -\boldsymbol{q}^2$$

Relation between scattering angles.

Let the particle m_1 moves in the Lab. along the x-axis with a velocity v_1 and m_2 is at rest in the Lab., $v_2 = 0$. In the CM frame let m_2 move with a velocity v_{20} . Thus the CM moves with respect to the Lab frame with a velocity v_{20} and in the x-direction i.e. in the same direction as the initial motion of m_1.

Thus by Lorentz transformations (25.2)

$$E_{10} = \gamma_2(E_1 - \beta_2 p_1) \,, \;\; p_{10} = \gamma_2(p_1 - \beta_2 E_1) \tag{25.45}$$

$$E_{20} = \gamma_2 E_2 \ , \qquad p_{20} = \gamma_2(-\beta_2 E_2)$$

Centre of mass frame is defined as

$$\boldsymbol{p}_{10} \times \boldsymbol{p}_{20} = 0$$

$$\therefore \quad \gamma_2 p_1 - \gamma_2 \beta_2 E_1 - \gamma_2 \beta_2 m_2 = 0$$

Thus the velocity of the CM frame w.r.t. the Lab frame is

$$\beta_2 = v_2 = \frac{p_1}{E_1 + m_2} \tag{25.46}$$

After scattering, the incident particle 1 has momentum p_1' which has components p_{1x}', p_{1y}' in the Lab frame and p_{10x}' and p_{10y}' in the C.M. frame

We have

$$\tan\theta_1 = \frac{p_{1y}'}{p_{1x}'} \tag{25.27}$$

Transforming from C.M. frame to Lab. $(\beta \to -\beta)$

$$p_{1x}' = \gamma_2'(p_{10x}' + \beta_2' E_{10}')$$

$$p_{1y}' = p_{10y}' \tag{25.48}$$

Since $\boldsymbol{p} = m\gamma\boldsymbol{v}$

$$p_{10x}' = m_1 v_{10}' \gamma_1' \cos\theta$$

$$p_{10y}' = m_1 v_{10}' \gamma_1' \sin\theta \tag{25.49}$$

$$E_{10}' = m_1 \gamma_1'$$

Also in the C.M. frame

$$p_{10}' = p_{20}' \text{ i.e. } m_1 v_{10}' \gamma_1' = m_2 v_{20}' \gamma_2'$$

$$\frac{\beta_2'}{\beta_1'} = \frac{v_{20}'}{v_{10}'} = \frac{m_1}{m_2}\frac{\gamma_1'}{\gamma_2'} \tag{25.50}$$

We thus get

$$\tan\theta_1 = \frac{m_1 v_{10}' \gamma_1' \sin\theta}{m_1 v_{10}' \gamma_1' \cos\theta + m_1 \gamma_1' \beta_2'}$$

$$= \frac{\sin\theta}{\gamma_2'\left(\cos\theta + \frac{\beta_2'}{\beta_1'}\right)}$$

$$= \frac{1}{\gamma_2'} \frac{\sin\theta}{\left(\cos\theta + \frac{m_1}{m_2}\frac{\gamma_1'}{\gamma_2'}\right)}$$

Can also be expressed as

$$\tan\theta_2 = \frac{p'\sin\theta}{\gamma(p'\cos\theta + v_2 E_C)} \qquad (25.51)$$

Where v_2 is given by (25.46) , $\gamma = (1 - v_2^2)^{-1/2}$

5. Summary:

❖ In special theory of Relativity, space and time are consider absolute and constitute coordinates in a 4 – D Minkowski space.

❖ The Lorentz transformations preserve the proper time element $ds^2 = d\tau^2 = dt^2 - d\boldsymbol{x}^2$ so that the velocity of light is independent of the inertial frame of reference.

❖ The scalar product of 4 – vectors is a Lorentz invariant whereas the 4 – vector V^μ itself transforms like the coordinates in an inertial coordinate system in Minkowski space.

$$dx'^\mu = \Lambda^\mu_\nu \; x^\nu$$

$$V'^\mu = \Lambda^\mu_\nu \; V^\nu$$

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$